

Phase transitions of orientifold gauge theories at large N in finite volume

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ABSTRACT: In this paper we consider the phase structure of “orientifold” gauge theories — obtained from unitary supersymmetric gauge theories by replacing adjoint Majorana fermions by Dirac fermions in the symmetric or anti-symmetric representations — in finite volume $\mathcal{S}^3 \times \mathcal{S}^1$. If the radius of the \mathcal{S}^3 is small the calculations can be performed at weak coupling for any value of the \mathcal{S}^1 radius. We demonstrate that there is a confinement/deconfining type of phase transition even when the fermions have periodic (non-thermal) boundary conditions around \mathcal{S}^1 . At small radius of \mathcal{S}^1 , the theory is in a phase where charge conjugation and large non-periodic gauge transformation are spontaneously broken. But for large radius of \mathcal{S}^1 the phase preserves these symmetries just as in the related supersymmetric theory.

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“Orientifold” gauge theories have been the subject of much interest due to the possibility of large N equivalence [1, 2]. The sobriquet “orientifold” comes from string theory, but field theorists may think of it as describing a gauge theory which has the same field content as a supersymmetric gauge theory but where one replaces the adjoint Majorana fermions by Dirac fermions transforming in either the anti-symmetric or symmetric representation.¹ Since the dimensions of the anti-symmetric and symmetric representations are $\frac{1}{2}N(N \pm 1)$ and that of the adjoint is N^2 , both theories have the same number of fermionic degrees-of-freedom in the large N limit. The idea is that certain observables of these theories will be equal to those of the original supersymmetric theory in the large N limit [3–6]. This is an intriguing possibility, but it is not easy to test the hypothesis because in \mathbf{R}^4 these theories are strongly coupled confining gauge theories. However, there are ways of rendering a strongly coupled theory weakly coupled and in the weakly coupled regime can attempt a test of the hypothesis. The first such proposal [7] considered the theory on $\mathbf{T}^3 \times \mathbf{R}$ and the conclusion was that for small tori needed to ensure weak coupling, large N equivalence was not manifest. A different kind of philosophy was proposed in [8] who considered the theory on $\mathbf{R}^3 \times \mathbf{S}^1$ with either periodic or anti-periodic (thermal) boundary conditions on the fermions. When the size of the circle (which we denote by β) is small, then the Wilson loop of the gauge field around the circle can get a large VEV $\sim \beta^{-1}$ which breaks the gauge group $U(N) \rightarrow U(1)^N$ at an energy scale much greater than Λ_{QCD} and so renders the theory weakly-coupled. An effective action for the eigenvalues of the Wilson loop, or Polyakov loop in the thermal case,

$$U = \text{Tr} \exp i \oint_{\mathbf{S}^1} A = \sum_{j=1}^N e^{i\theta_j}, \quad (1)$$

can be computed in this regime and minimized in order to find the ground state of the theory. The simplest kind of orientifold gauge theory involves in addition to the gauge field, a Dirac fermion transforming in the anti-symmetric or symmetric representation of the gauge group. The resulting ground state depends crucially on whether the boundary conditions on the fermions are periodic or anti-periodic. In the periodic case, the ground state has all the eigenvalues θ_i sitting at $\pi/2$ or all at $3\pi/2$ and the so there is spontaneous symmetry breaking of the \mathbf{Z}_2 large gauge transformations which take $\theta_i \rightarrow \theta_i + \pi$ (to be described more fully later) and also of charge conjugation which takes $U \rightarrow U^*$, or $\theta_i \rightarrow -\theta_i$.² For thermal boundary conditions, the ground state is either $\theta_i = 0$ or π and so, although \mathbf{Z}_2 is also broken, charge conjugation is preserved.

On the contrary, in the related “sister” theory, in this case $\mathcal{N} = 1$ Yang-Mills, with periodic boundary conditions for the fermions, the θ_i are always uniformly distributed around the circle and charge conjugation is always unbroken. This shows conclusively that large N equivalence is not valid on $\mathbf{R}^3 \times \mathbf{S}^1$ at small radius with periodic boundary conditions on the fermions. Unfortunately, one cannot say anything about large radius (and therefore

¹We are assuming a $U(N)$ or $SU(N)$ gauge group.

²In actual fact when all the eigenvalues are equal, the theory is not strictly speaking weakly coupled since there is no Higgs mechanism.

R^4) where the theory becomes strongly coupled using this approach. The problem is that there may be a phase transition in the orientifold theory for some critical radius for which Z_2 and charge conjugation are restored. Experience suggests that these kind of phase transitions do indeed occur in the thermal case with anti-periodic boundary conditions for the fermions. The phase transition in question is the confinement/de-confinement transition where the de-confined phase occurs at high temperature (small radius) and involves spontaneous breaking of large non-periodic gauge transformations and the generation of a condensate $\langle U \rangle \neq 0$. However, with periodic boundary conditions in the sister supersymmetric theory, this symmetry is preserved and the confinement/de-confinement transition does not occur. For the orientifold theory, however, we simply do not know. In the case of thermal (anti-periodic) boundary conditions on the fermions, [8] shows that at high temperature (small radius) both the orientifold and original theories break their group of large non-periodic gauge transformations, Z_2 and $U(1)$, respectively. The difference is that charge conjugation is preserved in both cases and so large N equivalence may be valid here.

There is a way of studying these kinds of phase transitions whilst remaining in a weakly-coupled regime. Namely, we can investigate the theory on $S^3 \times S^1$. In this situation, we can keep the radius of S^3 , $R \ll 1/\Lambda_{\text{QCD}}$, and then study the physics as a function of the ratio β/R for all values of β . Of course, one can, at the end of the day, argue that there are additional phase transitions as R varies but nevertheless the universality classes of the transitions seen at small R seem to match the expected transitions in the strongly coupled theories on R^3 at finite temperature. Our main conclusion is that in the orientifold case, there is a phase transition even with periodic boundary conditions.

We now turn to the calculations and, following the beautiful paper [9] whose results and notion we use extensively, we compute a Wilsonian effective action for the gauge theory on $S^3 \times S^1$ to the one loop order. The only zero modes belong to the constant mode of A_0 , the gauge field component around S^1 :

$$\alpha = \frac{1}{\text{Vol } S^3 \times S^1} \int_{S^3 \times S^1} A_0 . \tag{2}$$

We can use global gauge transformation to diagonalize α :

$$\alpha = \beta^{-1} \text{diag}(\theta_i) . \tag{3}$$

The θ_i are angular variables since there are large gauge transformations (but periodic around S^1) that take $\theta_i \rightarrow \theta_i + 2\pi$. Physically, the gauge invariant quantity is the Wilson loop (1). On top of this there are additional large gauge transformations that are only periodic on S^1 up to an element of the centre Γ of the gauge group a quantity that depends on the matter content. These large gauge transformation form a group themselves isomorphic to Γ and we will refer to it as $\tilde{\Gamma}$. If there is only adjoint matter and the gauge group is $SU(N)$ then $\tilde{\Gamma} = Z_N$, while for gauge group $U(N)$ we have $\tilde{\Gamma} = U(1)$. In the presence of matter the centre and hence $\tilde{\Gamma}$ can be a smaller subgroup of Z_N or $U(1)$, respectively. If there were only adjoint matter then in the $SU(N)$ case these non-periodic large gauge transformations take $\theta_i \rightarrow \theta_i + 2\pi/N$ and so transform U by an N -th root of unity. In the $U(N)$ case, the transformations are $\theta_i \rightarrow \theta_i + a$ for $0 \leq a < 2\pi$. Hence, strictly

speaking, the gauge invariant observables are, for example, $|U|$. Spontaneous symmetry breaking of this $\tilde{\Gamma}$ symmetry occurs when $\langle |U| \rangle \neq 0$.

The radiative corrections at the one loop level are obtained by taking the constant mode (3) as a background VEV and integrating out all the massive modes of the fields. The this end, we shift $A_0 \rightarrow A_0 + \alpha$ and then the one-loop contribution involves the logarithm of the resulting functional determinants and which depend on α in a non-trivial way.

The detailed calculations have been performed in [9] and so our discussion will be brief. Each field is expanded in terms of harmonics on $\mathbf{S}^3 \times \mathbf{S}^1$ and keeping only the quadratic terms and integrating out the fields, a typical contribution to the effective action is of the form

$$\pm \frac{1}{2} \text{Tr} \log(-\tilde{D}_0^2 - \Delta), \tag{4}$$

the ± 1 being for bosons and fermions, respectively. In the above, $\tilde{D}_0 = \partial_0 + i\alpha$ and so includes the coupling to the VEV, and Δ is the Laplacian on \mathbf{S}^3 appropriate to the tensorial nature of the field on \mathbf{S}^3 . The background VEV α acts as a generator of the Lie algebra of $SU(N)$ in the representation of the gauge group appropriate to the field and the trace includes a trace over that representation of the gauge group. The eigenvalues of ∂_0 are simply $2\pi i n/\beta$, $n \in \mathbf{Z}$, while the eigenvectors of the Laplacian on \mathbf{S}^3 are labelled by the angular momentum ℓ :

$$\Delta \psi_\ell = -\varepsilon_\ell^2 \psi_\ell, \tag{5}$$

and we denote their degeneracy as d_ℓ . The data ε_ℓ and d_ℓ depend on the field type as we list below.

- i) **Scalars.** There are two kinds of scalar fields which have the same set of eigenvectors. Firstly, for conformally coupled scalars³ we have $\varepsilon_\ell = R^{-1}(\ell + 1)$. On the other hand, for minimally coupled scalars $\varepsilon_\ell = R^{-1}\sqrt{\ell(\ell + 2)}$. Both types have a degeneracy $d_\ell = (\ell + 1)^2$ with $\ell \geq 0$.
- ii) **Spinors.** For 2-component complex spinors,⁴ we have $\varepsilon_\ell = R^{-1}(\ell + 1/2)$ and $d_\ell = 2\ell(\ell + 1/2)$ with $\ell > 0$.
- iii) **Vectors.** Here the situation is more complicated. A vector field V_i can be decomposed into the image and the kernel of the covariant derivative: $V_i = \nabla_i \chi + B_i$, with $\nabla^i B_i = 0$. The eigenvectors for the closed part, B_i , have $\varepsilon_\ell = R^{-1}(\ell + 1)$ and $d_\ell = 2\ell(\ell + 2)$ with $\ell > 0$. On the other hand, the exact part $\nabla_i \chi$ has $\varepsilon_\ell = R^{-1}\sqrt{\ell(\ell + 2)}$ with degeneracy $d_\ell = (\ell + 1)^2$ but with $\ell > 0$ only.

Notice that both spinors and vectors have no zero ($\ell = 0$) modes on \mathbf{S}^3 . This is why in a pure gauge theory the only field with a zero mode is A_0 which is a scalar on \mathbf{S}^3 .

³These have a mass term involving the Ricci scalar of the manifold, in this case simply R^{-1} .

⁴This representation is actually reducible on \mathbf{S}^3 into two real 2-component spinors but corresponds to a Majorana spinor on $\mathbf{S}^3 \times \mathbf{S}^1$.

It is a standard calculation using the identity $\prod_{n=1}^{\infty}(1+x^2/n^2) = \sinh(\pi x)/(\pi x)$ to show that (4) is equal, up to an infinite additive constant, to

$$\sum_{\ell=0}^{\infty} d_{\ell} \left\{ \beta \varepsilon_{\ell} - \sum_{n=1}^{\infty} \frac{1}{n} e^{-n\beta \varepsilon_{\ell}} \text{Tr} \cos(\beta \alpha) \right\}. \quad (6)$$

The first term here involves the Casimir energy and since it is independent of α will play no rôle in our story and we will subsequently drop it.

The sums over the angular momentum in (6) can be performed explicitly for each of the tensor types on \mathbf{S}^3 . Firstly for conformally coupled scalars

$$z_s(x) = \sum_{\ell=0}^{\infty} (\ell+1)^2 x^{-(\ell+1)} = \frac{x(1+x)}{(1-x)^3}, \quad (7)$$

whilst for spinors

$$z_f(x) = 2 \sum_{\ell=1}^{\infty} \ell(\ell+1/2) x^{-(\ell+1/2)} = \frac{4x^{3/2}}{(1-x)^3}, \quad (8)$$

and finally for closed vectors

$$z_v(x) = 2 \sum_{\ell=1}^{\infty} \ell(\ell+2) x^{-(\ell+1)} = \frac{x(6x-2x^2)}{(1-x)^3}, \quad (9)$$

where $x = e^{-\beta/R}$. We will not need the sums for minimally coupled scalars and exact vectors.

Before we consider theories with matter fields, let us first consider pure Yang-Mills where we have to face the issue of gauge fixing. One can follow the non-covariant gauge fixing in [9], however it is perhaps simpler to use a conventional Faddeev-Popov procedure and choose Feynman gauge. The gauge field A_{μ} includes A_0 which transforms as a minimally coupled scalar on \mathbf{S}^3 , while $A_i = B_i + \nabla_i \chi$. The ghosts transform as minimally coupled scalars on \mathbf{S}^3 but contribute with a -1 in (4) since they are Grassmann valued. The $\ell > 0$ contributions from A_0 , $\nabla_i \chi$ and the ghosts all cancel leaving only a net contribution from the $\ell = 0$ modes (since exact vector do not have an $\ell = 0$ mode) of the form

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{Tr} \cos(n\beta\alpha) = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{ij=1}^{\infty} \cos n(\theta_i - \theta_j). \quad (10)$$

This part is precisely the exponentiation of the Jacobian that converts the integrals over the θ_i into an integral over the unitary matrix $U = \text{diag}(e^{i\theta_i})$:

$$\int \prod_{i=1}^N d\theta_i \exp \left\{ \sum_{n=1}^{\infty} \frac{1}{n} \sum_{ij=1}^N \cos n(\theta_i - \theta_j) \right\} \propto \int \prod_{i=1}^N d\theta_i \prod_{i<j} \sin^2 \left(\frac{\theta_i - \theta_j}{2} \right) = \int dU. \quad (11)$$

However, we will leave the Jacobian in the exponent since it must be considered as part of the effective action for the eigenvalues.

The remaining modes are the closed vectors B_i and these contribute as in (6) with $d_\ell = 2\ell(\ell + 2)$ and $\varepsilon_\ell = R^{-1}(\ell + 1)$. Using the sum (9) and including the Jacobian term in (10), the full effective action is simply

$$S(\theta_i) = \sum_{n=1}^{\infty} \frac{1}{n} (1 - z_v(x^n)) \sum_{ij=1}^N \cos n(\theta_i - \theta_j) . \quad (12)$$

The phase structure as a function of the temperature is determined by minimizing $S(\theta_i)$.

As the temperature changes from low to high, the parameter x varies from 0 to 1. At low temperatures, the pre-factor of the cosine is positive and the eigenvalues effectively repel one another and for large N form a uniform distribution around the circle. As x increases the factor $1 - z_v(x)$ changes sign at some critical temperature $T = T_c$ which can be found by solving $z_v(e^{-1/RT_c}) = 1$. Beyond this the eigenvalues attract each other and in the limit of very high temperatures the distribution of eigenvalues becomes a delta function at some arbitrary point θ_0 around the circle. Notice that the transition is driven by the $n = 1$ term in (12). Of course in a situation with N finite there is no genuine symmetry breaking in finite volume and one should integrate over the modulus θ_0 . However, if we take the large N limit, then a sharp phase transition does indeed occur at T_c and the high temperature phase spontaneously breaks the $\tilde{\Gamma}$ symmetry.⁵ The order parameter is the expectation value of the Wilson/Polyakov loop with $\langle U \rangle = 0$ in the low temperature phase and $\langle U \rangle = N e^{i\theta_0} \neq 0$ in the high temperature phase. This pattern of symmetry breaking is precisely what one expects for the confinement/de-confinement phase transition in the strongly-coupled theory on \mathbf{R}^3 . Another order parameter is the effective action itself. At low temperatures $S = 0$ while above the transition $S = \mathcal{O}(N^2)$ as one would expect if the colour degrees-of-freedom were being de-confined.

Another way to analyze the effective action in the large N limit is to represent the distribution of the eigenvalues θ_i by a density $\rho(\theta)$ normalized so that $\int_0^{2\pi} d\theta \rho(\theta) = 1$ and replace $\sum_{i=1}^N f(\theta_i) \rightarrow N \int_0^{2\pi} d\theta \rho(\theta) f(\theta)$. To this end, it is useful to define the Fourier components

$$\rho_n^+ = \int_0^{2\pi} d\theta \rho(\theta) \cos(n\theta), \quad \rho_n^- = \int_0^{2\pi} d\theta \rho(\theta) \sin(n\theta), \quad (13)$$

with $\rho_0^+ = 1/(2\pi)$, in terms of which the effective action is

$$S(\rho_n^\pm) = \frac{N^2}{2\pi} \sum_{n=1}^{\infty} \{V_n^+(T)(\rho_n^+)^2 + V_n^-(T)(\rho_n^-)^2\}, \quad (14)$$

where, in this case,

$$V_n^+(T) = V_n^-(T) = \frac{2\pi}{n} (1 - z_v(x^n)) . \quad (15)$$

At low temperature, all the $V_n^\pm(T)$ are positive and so $\rho_n^\pm = 0$, $n > 0$. This means that only $\rho_0^+ = 1/(2\pi)$ is non-vanishing corresponding to the uniform distribution. As x increases

⁵This is $U(1)$ in the case of pure Yang-Mills with a $U(N)$ gauge group or if the gauge group is $SU(N)$ then $\theta_0 = 2\pi n/N$, $n \in \mathbf{Z}$, and $\tilde{\Gamma} = \mathbf{Z}_N$.

$V_1^\pm(T)$ change sign at the critical temperature and at $T = T_c$ the first harmonics can be non-vanishing corresponding to

$$\rho(\theta) = \frac{1}{2\pi}(1 + t \cos(\theta - \theta_0)) \tag{16}$$

for arbitrary t and θ_0 . At $T = T_c$, $0 \leq t \leq 1$ parameterizes a flat direction. Note that $V_n^\pm(T)$ for $n > 1$ change sign at a higher temperature and so the transition is determined solely by $V_1^\pm(T)$. For $T > T_c$ one can show that $\rho(\theta)$ develops a gap at $\theta = \theta_0 + \pi$ which grows as T increases so that at very high temperatures $\rho(\theta) = \delta(\theta - \theta_0)$. The order of the transition turns out to be a rather subtle issue that depends on higher orders in perturbation theory. Remarkably, the necessary three-loop calculation was performed for pure Yang-Mills in [10] where it was shown that the transition is first order.

$\mathcal{N} = 1$ Yang-Mills. In this theory, there is a adjoint-valued Majorana (or Weyl) fermion in addition to the gauge field. Now that the theory has fermions there are two possibilities for the boundary conditions of the fermions around the circle. In the thermal case, the fermions have anti-periodic boundary conditions and supersymmetry is broken while the other possibility is to have supersymmetry preserving periodic boundary conditions. The effective action is

$$S(\theta_i) = \sum_{n=1}^{\infty} \frac{1}{n} (1 - z_v(x^n) + \sigma_n z_f(x^n)) \sum_{ij=1}^N \cos n(\theta_i - \theta_j) . \tag{17}$$

where⁶

$$\sigma_n = \begin{cases} (-1)^n & \text{thermal (anti-periodic)} \\ 1 & \text{periodic} . \end{cases} \tag{18}$$

Or one can write it as (14) with

$$V_n^+(T) = V_n^-(T) = \frac{2\pi}{n} (1 - z_v(x^n) + \sigma_n z_f(x^n)) . \tag{19}$$

In this case the behaviour depends crucially on whether we have thermal or periodic boundary conditions for the fermions. In the thermal case, as T increases $V_1^\pm(T)$ change when $z_v(x) + z_f(x) = 1$ and there is a phase transition of exactly the same kind as in the pure gauge theory. On the other hand if we choose periodic boundary conditions then $V_n^\pm(T)$ are always positive and no transition occurs. In this case the system is always in the state with a uniform distribution of eigenvalues.

The orientifold theory. This theory has a Dirac fermion in the anti-symmetric or symmetric representation⁷ as well as the gauge field. In this case, with gauge group $U(N)$, the group of large non-periodic gauge transformations is $\tilde{\Gamma} = \mathbf{Z}_2$.

⁶This can be derived by shifting the Matsubara frequencies $2\pi n/\beta$ by π/β in the anti-periodic case.

⁷This is equivalent to having one Weyl in the anti-symmetric and one in its complex conjugate representation.

In the anti-symmetric representation, the eigenvalues of α are $\theta_i + \theta_j$, $i < j$, along with $-\theta_i - \theta_j$, $i < j$, for its conjugate.⁸ The group $\tilde{\Gamma} = \mathbf{Z}_2$ is generated by $\theta_i \rightarrow \theta_i + \pi$. Using all the formulae above, one finds that the the effective action is⁹

$$S(\theta_i) = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{i \neq j=1}^N \left\{ (1 - z_v(x^n)) \cos n(\theta_i - \theta_j) + \sigma_n z_f(x^n) \cos n(\theta_i + \theta_j) \right\}, \quad (20)$$

Before we consider the large N limit, it is quite instructive to consider the case of $N = 2$. As in other cases, changes of state are driven solely by the $n = 1$ terms which in this case are

$$S(\theta_i) = 2(1 - z_v(x)) \cos(\theta_1 - \theta_2) + 2\sigma_1 z_f(x) \cos(\theta_1 + \theta_2) + \dots . \quad (21)$$

It is a simple matter to minimize these $n = 1$ terms with respect to θ_1 and θ_2 . Notice that $z_f(x)$ is always positive, whereas $1 - z_v(x)$ is positive at low temperature and negative at high temperature. In the thermal case, $\sigma_1 = -1$ and therefore the low temperature phase has eigenvalues $(\pi/2, 3\pi/2)$. There is a transition at $z_v(x) + z_f(x) = 1$ and in the high temperatures phase one has the two states $(0, 0)$ or (π, π) . Of course, since we are in finite volume we have to sum over the two high temperature states and the \mathbf{Z}_2 symmetry is restored. However, below we shall consider the large N limit where there is a genuine sharp phase transition and symmetry breaking.

Now we turn to the case of periodic boundary conditions for which $\sigma_1 = 1$. In this case, at low temperature we have the unique state $(0, \pi)$ but there is also a transition when $z_v(x) + z_f(x) = 1$. The high temperature phase has two solutions $(\pi/2, \pi/2)$ or $(3\pi/2, 3\pi/2)$. The pattern of eigenvalues matches those found for the theory on $\mathbf{R}^3 \times \mathbf{S}^1$ in [8].

While the finite N case is interesting, in order to have a genuine phase transition we need to extend the discussion to the large N limit. In this case, we find

$$V_n^{\pm}(T) = \frac{2\pi}{n} (1 - z_v(x^n) \pm \sigma_n z_f(x^n)) . \quad (22)$$

At low temperatures both $V_n^{\pm}(T)$ are positive for both thermal and periodic boundary conditions. In this case, the ground state consists of a uniform distribution of eigenvalues $\rho(\theta) = 1/(2\pi)$. As T increases there is a transition at $z_v(x) + z_f(x) = 1$ where $V_1^+(T)$ changes sign, for thermal boundary conditions. This is exactly the same critical temperature as in the $\mathcal{N} = 1$ supersymmetric theory. In this case $V_1^-(T)$ remains positive. The situation for periodic boundary conditions is the reverse of this: $V_1^-(T)$ changes sign and $V_1^+(T)$ remains positive. Notice that the critical temperature is the same in both cases. At the critical temperature the distribution can develop the $\cos \theta$ harmonic, in the thermal case, and the $\sin \theta$ harmonic, in the periodic case. In the high temperature phase the \mathbf{Z}_2

⁸In the symmetric representation the eigenvalues include the same but in addition the ones with $i = j$. For the most part we shall consider the anti-symmetric representation but at large N the conclusions for the symmetric representation will be identical.

⁹Note that we have restricted the sum to $i \neq j$ for the anti-symmetric representation, however, this has no effect on the terms arising from the gauge field which are independent of the eigenvalues when $i = j$.

symmetry $\theta \rightarrow \theta + \pi$ is spontaneously broken with the limiting distributions $\rho(\theta) = \delta(\theta)$ or $\delta(\theta - \pi)$, in the thermal case, and $\delta(\theta - \pi/2)$ and $\delta(\theta - 3\pi/2)$, in the periodic case. Notice that in the periodic case, charge conjugation symmetry $\rho(\theta) \rightarrow \rho(-\theta)$ is also spontaneously broken, but this does not occur in the thermal case.

In the periodic case, the two phases match precisely those found in [8] for the theory on $\mathbf{R}^3 \times \mathbf{S}^1$ for small radius. However, as the \mathbf{S}^1 in our case de-compactifies, we find a phase transition to the uniform distribution of eigenvalues just as in the sister supersymmetric theory. This does not prove definitively that such a transition will also occur in the theory on $\mathbf{R}^3 \times \mathbf{S}^1$ since this is a strongly-coupled theory and perturbative or semi-classical methods are not valid.

Notice that the same formalism is also applicable to the case of the symmetric representation by simply extending the sums in (20) to include the $i = j$ terms. These terms are sub-leading at large N and so do not affect the conclusions.

The orientifold $\mathcal{N} = 4$ theory. This theory has the same matter content as the $\mathcal{N} = 4$ theory apart from the fact that the four adjoint Majorana fermions are replaced by Dirac fermions in the symmetric or anti-symmetric representation of the gauge group [12].

For the $\mathcal{N} = 4$ theory which has, in addition to the gauge field 6 conformally coupled scalars and 4 Majorana fermions all transforming in the adjoint representation of the gauge group, we have

$$V_n^+(T) = V_n^-(T) = \frac{2\pi}{n} (1 - z_v(x^n) - 6z_s(x^n) + 4\sigma_n(x)z_f(x^n)) . \quad (23)$$

In the thermal case, there is a phase transition when $z_v(x) + 6z_s(x) + 4z_f(x) = 1$ in the same universality class as in the pure Yang-Mills case discussed above. In the periodic case, $1 - z_v(x) - 6z_s(x) + 4z_f(x)$ is always positive and no phase transition occurs.

For the orientifold version, the fermions come in either the symmetric or anti-symmetric representation, and in these cases,

$$V_n^\pm(T) = \frac{2\pi}{n} (1 - z_v(x^n) - 6z_s(x^n) \pm 4\sigma_n(x)z_f(x^n)) . \quad (24)$$

Hence in both the thermal and periodic cases there is a phase transition at $z_v(x) + 6z_s(x) + 4z_f(x) = 1$ in the same universality classes as the orientifold theory described above. So in the thermal case, in the high temperature case \mathbf{Z}_2 is spontaneously broken, but not charge conjugation, whilst in the periodic case both \mathbf{Z}_2 and charge conjugation are spontaneously broken.

It would be interesting to relate this weak coupling picture to the gravity dual. For the $\mathcal{N} = 4$ theory, Witten argued that the phase transition at strong coupling in the thermal case, should correspond to the Hawking-Page transition between thermal AdS (Euclidean AdS_5 with a periodic identification) and the Euclidean AdS big black hole [11]. The big black hole describes the de-confined phase at high temperature. Both geometries have a one cycle on the boundary, however, in the black hole case the cycle is contractable as one goes into the interior forming a cigar in the bulk geometry. In thermal AdS the one cycle is not contractable. These features match the fact that in the high temperature phase there

is a non-vanishing Polyakov loop at weak coupling which in the gravity dual arises from a string world-sheet that wraps the cigar. For thermal AdS, there is no contractable one cycle matching the fact that the Polyakov loop vanishes in the low temperature phase. Another symptom of the fact that the one cycle on the boundary of the black hole is contractable, is that only fermions with anti-periodic boundary conditions can be supported. Hence there is no Hawking-Page transition for periodic fermion boundary conditions matching perfectly with the weak coupling picture.

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References

- [1] S. Kachru and E. Silverstein, *4D conformal theories and strings on orbifolds*, *Phys. Rev. Lett.* **80** (1998) 4855 [[hep-th/9802183](#)].
- [2] M.J. Strassler, *On methods for extracting exact non-perturbative results in non-supersymmetric gauge theories*, [hep-th/0104032](#).
- [3] A. Armoni, M. Shifman and G. Veneziano, *Exact results in non-supersymmetric large- N orientifold field theories*, *Nucl. Phys.* **B 667** (2003) 170 [[hep-th/0302163](#)].
- [4] A. Armoni, M. Shifman and G. Veneziano, *SUSY relics in one-flavor QCD from a new $1/N$ expansion*, *Phys. Rev. Lett.* **91** (2003) 191601 [[hep-th/0307097](#)].
- [5] A. Armoni, M. Shifman and G. Veneziano, *From super-Yang-Mills theory to QCD: planar equivalence and its implications*, [hep-th/0403071](#).
- [6] A. Armoni, M. Shifman and G. Veneziano, *QCD quark condensate from SUSY and the orientifold large- N expansion*, *Phys. Lett.* **B 579** (2004) 384 [[hep-th/0309013](#)].
- [7] J.L.F. Barbon and C. Hoyos, *Small volume expansion of almost supersymmetric large- N theories*, *JHEP* **01** (2006) 114 [[hep-th/0507267](#)].
- [8] M. Unsal and L.G. Yaffe, *(In)validity of large- N orientifold equivalence*, *Phys. Rev.* **D 74** (2006) 105019 [[hep-th/0608180](#)].
- [9] O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas and M. Van Raamsdonk, *The Hagedorn/deconfinement phase transition in weakly coupled large- N gauge theories*, *Adv. Theor. Math. Phys.* **8** (2004) 603 [[hep-th/0310285](#)].
- [10] O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas and M. Van Raamsdonk, *A first order deconfinement transition in large- N Yang-Mills theory on a small S^3* , *Phys. Rev.* **D 71** (2005) 125018 [[hep-th/0502149](#)].
- [11] E. Witten, *Anti-de Sitter space, thermal phase transition and confinement in gauge theories*, *Adv. Theor. Math. Phys.* **2** (1998) 505 [[hep-th/9803131](#)].
- [12] C. Angelantonj and A. Armoni, *Non-tachyonic type 0B orientifolds, non-supersymmetric gauge theories and cosmological RG flow*, *Nucl. Phys.* **B 578** (2000) 239 [[hep-th/9912257](#)].